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Spectral discontinuities in constrained dynamical models

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Abstract

As examples of models having interesting constraint structures, we derive a quantum mechanical model from the spatial freezing of a well-known relativistic field theory—the chiral Schwinger model. We apply the Hamiltonian constraint analysis of Dirac (1964 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science)) and find that the nature of constraints depends critically on a c -number parameter present in the model. Thus, a change in the parameter alters the number of dynamical modes in an abrupt and non-perturbative way. We have obtained new *real* energy levels for the quantum mechanical model as we explore *complex* domains in the parameter space. These were forbidden in the parent chiral Schwinger field theory where the analogue Jackiw–Rajaraman parameter is restricted to be real. We explicitly show the existence of modes that satisfy the higher derivative Pais–Uhlenbeck form of dynamics (Pais and Uhlenbeck 1950 *Phys. Rev.* **79** 145). We also show that the Cranking model (Valatin 1956 *Proc. R. Soc. A* **238** 122), well known in nuclear physics, can be interpreted as a spatially frozen version of another well-studied relativistic field theory in $(2 + 1)$ -dimensions—the Maxwell–Chern–Simons–Proca model (Deser *et al* 1982 *Phys. Rev. Lett.* **48** 975, Deser *et al* 1982 *Ann. Phys.* **140** 372).

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Hamiltonian formulation of constrained dynamical systems, as formulated by Dirac [1], provides a systematic framework to analyze and quantize constrained systems. In this scheme, there is an extremely important classification of constraints: first class constraints (FCCs) and

second class constraints (SCCs) (for a brief discussion, see section 2) and these two types of constraints act in qualitatively different manners. In a particular model with constraints, it might happen that as one moves smoothly in the parameter space the nature of the constraint system changes from FCC to SCC. Clearly, this will lead to a dramatic (and abrupt) change in the spectra and dynamics even though the model (Lagrangian or Hamiltonian) itself will not show any drastic change (such as the appearance of an explicit singularity or otherwise). What we mean by this behavior will become clear later when we discuss a specific model. This is quite in contrast to the normal behavior of a system toward a change in its parameters where small changes in the parameter are reflected in a small change in the dynamics. The reason, in the present case, is that the change in the parameter value is associated with a change in the constraint structure governing the system. The change in the nature of the constraints induces a non-perturbative change in the entire system. In physical terms, as the constraint structure changes from SC to FC, the system gains more symmetry (in the form of local gauge invariance) and this is reflected in a more restricted dynamics in the FCC system than in the SCC system. This phenomenon can be observed in a generic model where a physical and dynamical mode present in the SCC system abruptly vanishes at that particular point in the parameter space where the system becomes FCC. This is nicely revealed in the specific example we provide in this paper.

In this paper, we will demonstrate that in both the particle model and its parent field theoretic model, an entire Harmonic oscillator mode (in the former) and its relativistic field theoretic analogue—a massive Klein–Gordon mode (in the latter)—disappear as one passes from the SCC to the FCC system. We stress that this passage in the parameter space is smooth as far as the explicit expressions of the Lagrangian or the Hamiltonian of the model are concerned, and the non-perturbative changes in spectra and dynamics are seen only after a proper constraint analysis of the systems.

Moreover, we will also study the particle model for complex values of the parameter and show that it can yield physical (real) energy states which reminds us of similar behavior in non-Hermitian PT -symmetric models [2]. In the field theoretic example, we will show that at the crossover point in the parameter space the dynamics is governed by a higher order dynamical equation which can be thought of as a field theory analogue of the Pais–Uhlenbeck oscillator [3] of revived interest [4].

The paper is structured as follows. In section 2, we give a brief discussion on the constraints that are relevant for this paper. In section 3, we study the model that has constraints and the constraint classification depends on a parameter value present in the model. We analyze this singular point in parameter space in detail. Next, we move on to complex values of the parameter and obtain some hitherto unknown results for this particular model. This finite dimensional model has been derived from a very well-known field theory model, the bosonized chiral Schwinger model [5, 6], that we study briefly in section 4. In fact, the connection with the Pais–Uhlenbeck model will be made in this section. In section 5, we provide another example of a similar identification between well-studied finite dimensional and field theory models: the Cranking model [7] and the Maxwell–Chern–Simons–Proca (MCSP) field theory [8–10]. We end with concluding remarks in section 6.

2. Constraint analysis

In the Hamiltonian formulation of a constrained system [1] any relation between dynamical variables, not involving time derivative, is considered as a constraint. Constraints can appear from the construction of the canonically conjugate momenta (known as the primary constraint) or they can appear from demanding time invariance of the constraints (secondary constraint).

Once the full set of constraints is in hand, they are classified as FCC or SCC according to whether the constraint Poisson bracket algebra is closed or not, respectively. The presence of constraints indicates a redundance of degrees of freedom (DOF) so that not all the DOFs are independent. FCCs signal the presence of local gauge invariances in the system. If FCCs are present, there are two ways of dealing with them as follows: either one keeps all the DOFs but imposes FCCs by restricting the set of physical states to those satisfying the $\langle \text{FCC} | \text{state} \rangle = 0$ or one is allowed to choose further constraints, known as gauge fixing conditions, so that these together with FCCs turn into an SCC set and also give rise to Dirac brackets that we presently discuss. In the case of two SCCs, say $(\text{SCC})_1$ and $(\text{SCC})_2$ with the Poisson bracket $\{(\text{SCC})_1, (\text{SCC})_2\} = \mathcal{C}$ where $\mathcal{C} \neq 0$ is not another constraint, proceeding as before with $\langle \text{SCC} | \text{state} \rangle = 0$ one reaches an inconsistency because in the identity $\langle \text{state} | \{(\text{SCC})_1, (\text{SCC})_2\} | \text{state} \rangle = \langle \text{state} | \mathcal{C} | \text{state} \rangle$ the lhs = 0 but rhs $\neq 0$. For consistent imposition of SCCs, one defines the Dirac brackets between two generic variables A and B :

$$\{A, B\}_{\text{DB}} = \{A, B\} - \{A, (\text{SCC})_i\} \{(\text{SCC})_i, (\text{SCC})_j\}^{-1} \{(\text{SCC})_j, B\}, \quad (1)$$

where $(\text{SCC})_i$ is a set of SCCs and $\{(\text{SCC})_i, (\text{SCC})_j\}$ is the constraint matrix. For SCCs this matrix is invertible (for a finite dimensional bosonic system, the number of SCCs is always even) and since $\{A, (\text{SCC})_i\}_{\text{DB}} = \{(\text{SCC})_i, A\}_{\text{DB}} = 0$ for all A , one can implement $\text{SCC}_i = 0$ which strongly means that some of the DOFs can be removed thereby reducing the number of DOFs in the system but one must use the Dirac brackets in all subsequent computations. Hence, SCCs induce a change in the symplectic structure and subsequently one quantizes the Dirac brackets. The same principle is valid for a gauge-fixed FCC system mentioned before. Hence, to understand the effect of constraints we note that the presence of one FCC (together with its gauge fixing constraint) or SCC can remove two or one DOFs from the phase space respectively. We will apply this scheme in a specific model.

3. Particle model

Let us consider the following Lagrangian:

$$L = \frac{1}{2} \dot{A}_1^2 + \frac{1}{2} \dot{\phi}^2 + e(A_0 \dot{\phi} + \dot{A}_1 \phi) + \frac{ae^2}{2} (A_0^2 - A_1^2), \quad (2)$$

where an overdot represents the derivatives with respect to time and a and e are numerical parameters. A_0, A_1, ϕ constitute the dynamical variables. The somewhat unconventional notation will become clear when we connect this model with the field theory [6], where $A_i, i = 0, 1$, and ϕ will become electromagnetic gauge potentials and a scalar field respectively with e being the electric charge. Hence, although not mandatory, we prefer to keep e unchanged and explore the parameter space by varying a .

The conjugate momenta are $\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} + eA_0, \pi^1 = \frac{\partial L}{\partial \dot{A}_1} = \dot{A}_1 + e\phi, \pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0$, and one immediately notes a primary constraint

$$\psi_1 \equiv \pi_0 \approx 0. \quad (3)$$

The Hamiltonian is computed as

$$H = \pi \dot{\phi} + \pi^0 \dot{A}_0 + \pi^1 \dot{A}_1 - L = \frac{1}{2} [(\pi_1 + e\phi)^2 + (\pi - eA_0)^2 - ae^2 (A_0^2 - A_1^2)] + \lambda \pi_0, \quad (4)$$

where we append the constraint ψ_1 through a Lagrange multiplier. The canonical Poisson brackets are

$$\{A_\mu, \pi^\nu\} = g_\mu^\nu, \quad \{\phi, \pi\} = 1.$$

We use the metric $g_{\mu\nu} = \text{diag}(1, -1)$ and $\epsilon^{01} = 1$.

The time persistence of the primary constraint ψ_1 leads to the secondary constraint

$$\psi_2 \equiv \dot{\psi}_1 = \{\pi_0, H\} = \pi + e(a - 1)A_0 \approx 0. \quad (5)$$

Note that there are no further constraints since $\dot{\psi}_2$ will not yield a new constraint but only fix the Lagrange multiplier λ . This happens because the pair ψ_i, ψ_2 is SCC as we find out below.

The non-vanishing constraint bracket,

$$\{\psi_1, \psi_2\} = -e(a - 1), \quad (6)$$

shows that for $a \neq 1$ the set ψ_i is SCC. Clearly this is an explicit example of the interesting scenario that we mentioned earlier because for $a = 1$, this set is not SCC and in fact we will later show that actually $\pi_0 = 0$ turns out to be FCC for the special case $a = 1$. However, there is clearly no significant qualitative change or singular behavior in the *explicit expression* of the Lagrangian (2) or the Hamiltonian (4) as we pass through the point $a = 1$ in the parameter space for positive a .

3.1. $a \neq 1$

We now stick to $a \neq 1$ and compute the constraint matrix along with its inverse:

$$\{\psi_i, \psi_j\} = \begin{pmatrix} 0 & -e(a - 1) \\ e(a - 1) & 0 \end{pmatrix}, \quad \{\psi_i, \psi_j\}^{-1} = \begin{pmatrix} 0 & \frac{1}{e(a-1)} \\ -\frac{1}{e(a-1)} & 0 \end{pmatrix}. \quad (7)$$

Use of (1) leads to the Dirac brackets

$$\{A_0, \phi\}_{\text{DB}} = \frac{1}{e(a - 1)}, \quad \{A_1, \pi_1\}_{\text{DB}} = -1, \quad \{\phi, \pi\}_{\text{DB}} = 1 \quad (8)$$

and the reduced Hamiltonian

$$H = \frac{1}{2} \left[(\pi_1 + e\phi)^2 + ae^2 A_1^2 + \frac{a}{a - 1} \pi^2 \right]. \quad (9)$$

Note that the singular behavior at $a \rightarrow 1$ is now manifest in the Dirac brackets (8) or the reduced Hamiltonian (9), but it is only after we have properly taken care of the constraints.

However, it is crucial to note that the $a = 1$ point has to be considered separately since the constraints ψ_i commute, i.e. $\{\psi_1, \psi_2\} = 0$, and hence FCCs and the system have to be dealt in a completely different way to which we will come later. But for now, suffice it to say that $a = 1$ has a special significance.

From the following equations of motion $\dot{\phi} = \{\phi, H\}_{\text{DB}} = \frac{a}{a-1}\pi$, $\dot{\pi} = -e(\pi_1 + e\phi)$, $\dot{A}_1 = -(\pi_1 + e\phi)$, $\dot{\pi}_1 = ae^2 A_1$, we recover the spectra

$$(\pi_1 + e\phi)^\cdot = -\frac{a^2 e^2}{a - 1} (\pi_1 + e\phi), \quad (\pi - eA_1)^\cdot = 0. \quad (10)$$

From the bracket $\{(\pi - eA_1), (\pi_1 + e\phi)\}_{\text{DB}} = 0$, we find that the model is, in fact, free. For convenience, we rename the variables: $A_1 = x_1$, $\phi = x_2$, $\pi_1 = p_1 - e\phi = p_1 - ex_2$, $\pi = p_2 + eA_1 = p_2 + ex_1$ such that (x_1, p_1) and (x_2, p_2) constitute two independent canonical pairs. The Hamiltonian and dynamical equations are

$$H = \frac{1}{2} \left[p_1^2 + p_2^2 + \frac{a^2 e^2}{a - 1} \left(x_1 + \frac{p_2}{ae} \right)^2 \right], \quad (11)$$

$$\dot{p}_2 = 0, \quad \ddot{p}_1 = -\frac{a^2 e^2}{a - 1} p_1 \equiv -\omega^2 p_1, \quad \ddot{x}_1 = -\omega^2 x_1. \quad (12)$$

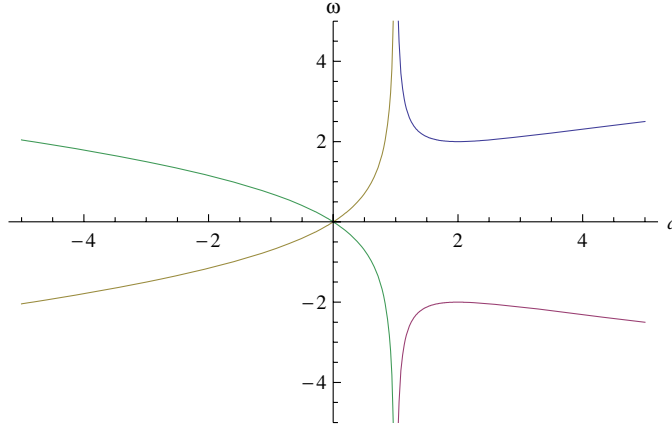


Figure 1. Variation of ω against a (a is real). The blue and red lines correspond to the positive real part and negative real part of ω , respectively; the green and yellow lines correspond to the positive imaginary part and negative imaginary part of ω , respectively. Here, the energy ω has bound $|\text{Re}[\omega]| \geq 2$.

Clearly, we dealing with a Harmonic oscillator (HO) and a decoupled free particle. Since p_2 is a constant with a suitable boundary condition, we put $p_2 = 0$ and end up with $H = \frac{1}{2}[p_1^2 + \frac{a^2 e^2}{a-1} x_1^2]$. Hence, we have an HO with frequency ω satisfying $\omega^2 = \frac{a^2 e^2}{a-1}$.

Note that the HO frequency or equivalently the quantized HO energy becomes complex for $a < 1$ and is real for $a > 1$. Apparently, the energy diverges for $a = 1$ signaling a singularity. However, as we have already mentioned, we have to treat the $a = 1$ case separately since the constraint structure shifts from SCC to FCC for $a = 1$. We show that the theory is indeed regular at $a = 1$ but with a different spectrum.

3.2. $a = 1$

Let us return to the starting Lagrangian (2) and directly put $a = 1$:

$$L = \frac{1}{2} \dot{A}_1^2 + \frac{1}{2} \dot{\phi}^2 + e A_0 \dot{\phi} + e \phi \dot{A}_1 + \frac{1}{2} e^2 (A_0^2 - A_1^2). \quad (13)$$

Similar analysis as before now yields the momenta $\pi = \dot{\phi} + e A_0, \pi^0 = 0, \pi^1 = \dot{A}_1 + e \phi$ and the Hamiltonian

$$H = \frac{1}{2} \pi^2 + \frac{1}{2} e^2 \phi^2 + \frac{1}{2} e^2 A_1^2 + \frac{1}{2} \pi_1^2 + e \phi \pi_1 - e \pi A_0. \quad (14)$$

Now we have three constraints: $\chi \equiv \pi_0 \approx 0, \psi_1 \equiv \dot{\chi} = \{\chi, H\} = \pi \approx 0, \psi_2 \equiv \dot{\psi}_1 = \{\psi_1, H\} = \pi_1 + e \phi \approx 0$, where χ is FCC since it commutes with all constraints, i.e. $\{\chi, \psi_i\} = 0$, and ψ_i are SCCs since $\{\psi_1, \psi_2\} = -e$. Presence of FCC χ allows us to choose another constraint $G \equiv A_0 = 0$ as a gauge condition such that the set $\{G, \chi\} = 1$ becomes a set of SCCs. Hence in this particular gauge, we have a completely SCC system of four constraints. Now the Dirac brackets and the reduced Hamiltonian H are $\{A_1, \phi\}_{\text{DB}} = \frac{1}{e}, \{A_1, \pi_1\}_{\text{DB}} = -1, H = \frac{1}{2} e^2 A_1^2$. Re-scaling the variables, $-e A_1 \equiv p, \frac{1}{e} \pi_1 \equiv x$, we find just the trivial dynamics of a free particle:

$$H = \frac{p^2}{2}, \quad \ddot{x} = 0. \quad (15)$$

Let us pause to observe that the spectra (15) (for $a = 1$) are very different from the previous case (12) (for $a \neq 1$). We find that the HO excitation is absent for $a = 1$ due to additional (gauge) symmetry in the system that further restricts the dynamical content. Physically as $a \rightarrow 1$, the HO frequency diverges so that its motion averages out and the excitation drops out from the spectrum. But this amounts to a form of coalescence of energy levels because as one approaches $a = 1$ from $a < 1$ (see figure 1), it appears that the imaginary part of the frequencies $\omega_{\pm} = \pm i \frac{ae}{\sqrt{a-1}}$ will end up at $\mp\infty$ respectively (and hence will not meet) but that is not the case since at $a = 1$ the HO energy is effectively zero. Hence, the full tower of HO states coalesces to the ground state (we will comment on this at the end).

In all the figures, we have taken $e = 1$.

3.3. Complex a

As we will mention in section 4, in the field theoretic model [6] a is a real number but for the present particle model we are free to consider complex $a = a_0 + ia_1$ and *a priori* we have a non-Hermitian model in (2). The explicit form of energy for complex a is

$$\omega = \pm e \left(\left(\sqrt{(a_0 - 1)^2 + a_1^2} + 1 \right) \frac{\left(\sqrt{(a_0 - 1)^2 + a_1^2} + a_0 - 1 \right)^{\frac{1}{2}}}{\left(2 \left((a_0 - 1)^2 + a_1^2 \right) \right)^{\frac{1}{2}}} + i \left(\sqrt{(a_0 - 1)^2 + a_1^2} - 1 \right) \frac{\left(\sqrt{(a_0 - 1)^2 + a_1^2} - a_0 + 1 \right)^{\frac{1}{2}}}{\left(2 \left((a_0 - 1)^2 + a_1^2 \right) \right)^{\frac{1}{2}}} \right). \quad (16)$$

For $a_1 = 0$, it reduces to the previous one (12). However, we find new results by requiring ω to be real even with complex a ($a_1 \neq 0$). We find two interesting consequences as follows. (i) There is a non-trivial relation giving rise to the separate bounds:

$$(a_0 - 1)^2 + a_1^2 = 1; \quad |a_1| < 1, \quad |a_0 - 1| < 1. \quad (17)$$

Using (17), we find that ω reduces to real values $\omega = \pm e\sqrt{2a_0}$ and *the singularity of ω disappears*. Furthermore, we can have real and positive energies even for $|a| = \sqrt{(a_0)^2 + (a_1)^2} < 1$. This can be contrasted with real a where only $a > 1$ will yield positive and real energy as in (12). (ii) For $a_1 \neq 0$, due to relation (17), a restriction is imposed on the real part a_0 . Again for real a , there appears no such restriction apart from $a > 1$ as in (12).

3.4. Discussion of figures 2–5

For figures 2–5, we have taken the positive part of ω from expression (16). Similar analysis could be done with the negative part. In figures 2 and 3, we plot $\text{Re}[\omega]$ versus $a_0(a_1)$ keeping $a_1(a_0)$ fixed. In figure 2, note that $a_1 = 0$ (blue line) is singular at $a_0 = 1$ (as in figure 1) but for non-zero a_1 values the lines are not singular near $a_0 = 1$. In figure 3, $a_0 = 1$ (yellow line) also diverges at $a_1 = 0$ since this corresponds to $a_0 = 1$ from (17).

More interesting features are found in figures 4 and 5, where we plot $\text{Im}[\omega]$ versus $a_0(a_1)$ keeping $a_1(a_0)$ fixed. We point out that when a_0 and a_1 obey relation (17), $\text{Im}[\omega]$ falls to zero giving real values of energy. Note that for complex a , the previous singularity (see in figures 4

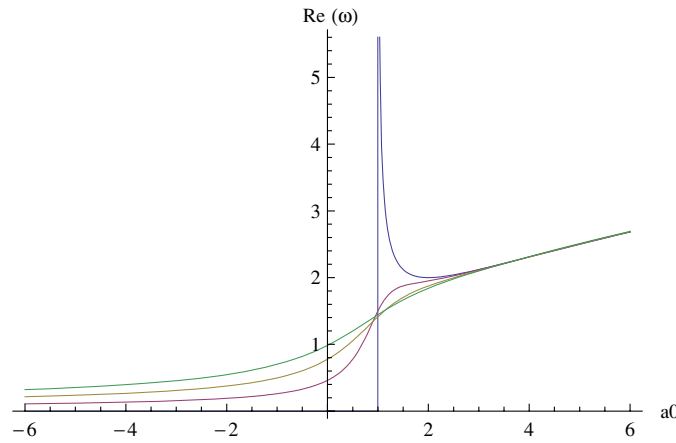


Figure 2. Variation of the positive value of $\text{Re}[\omega]$ against a_0 (keeping a_1 fixed). The different values of a_1 are 0 (blue), 0.5 (red), 1 (yellow) and 1.5 (green).

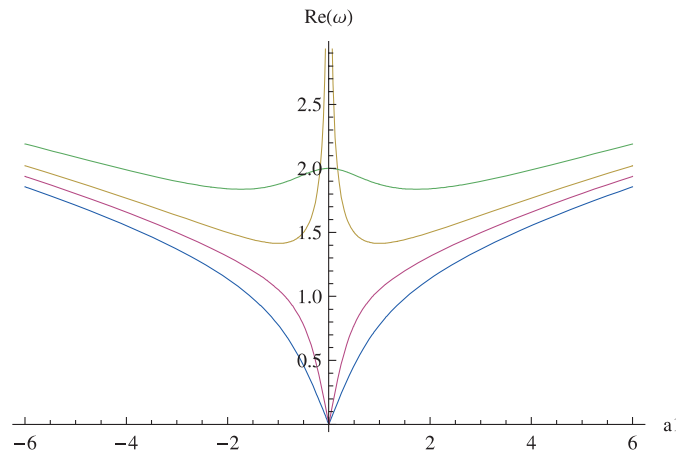


Figure 3. Variation of the positive value of $\text{Re}[\omega]$ against a_1 (keeping a_0 fixed). The different values of a_0 are 0 (blue), 0.5 (red), 1 (yellow) and 2 (green).

and 5) at $a = 1$ disappears. Furthermore, we show that it is possible to have real energy even for $|a| < 1$ whereas for real a , real energy is possible only for $a > 1$ (see figure 1).

In figure 4, the expression for $\text{Im}[\omega]$ is symmetric in a_1 and for $|a_1| > 1$, $\text{Im}[\omega]$ (green line) never reaches zero.

In figure 5, for $a_0 < 0$ the lines are entirely in positive $\text{Im}[\omega]$ sides and never vanish as they are outside the bounds in (17). But for $a_0 > 2$, $\text{Im}[\omega]$ will vanish once at $a_1 = 0$ which belongs to the normal behavior and so is not shown in the figure.

We now show that for complex a , energy can be real and $\omega < 2$ in contrast to the case for real a where $\omega \geq 2$. As an example, we put $\omega = 1$ in the energy expression (12). It gives $a = (1 \pm i\sqrt{3})/2$ and from figure 5 we find that for $a_0 = 0.5$, $\text{Im}[\omega] = 0$ (yellow line) at $a_1 = \pm\sqrt{3}/2$. Also from figure 3 for $a_0 = 0.5$ (red line), we find $\text{Re}[\omega] = 1$ for $a_1 = \pm\sqrt{3}/2$. In fact for $2 > a_0 > 0$, all the states having energy < 2 are now allowed.

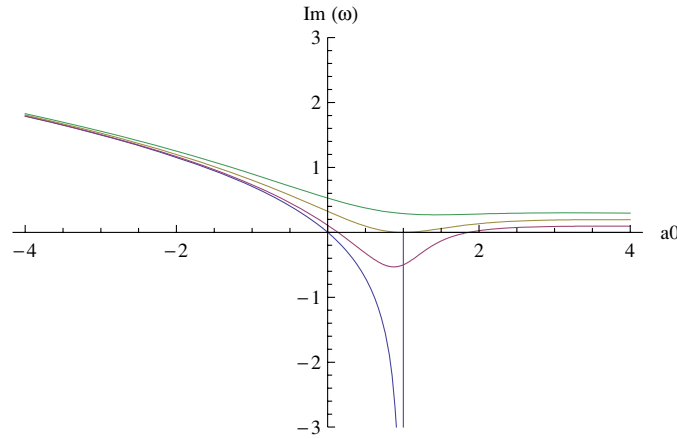


Figure 4. Variation of the positive value of $\text{Im}[\omega]$ against a_0 (keeping a_1 fixed). The different values of a_1 are 0 (blue), 0.5 (red), 1 (yellow) and 1.5 (green).

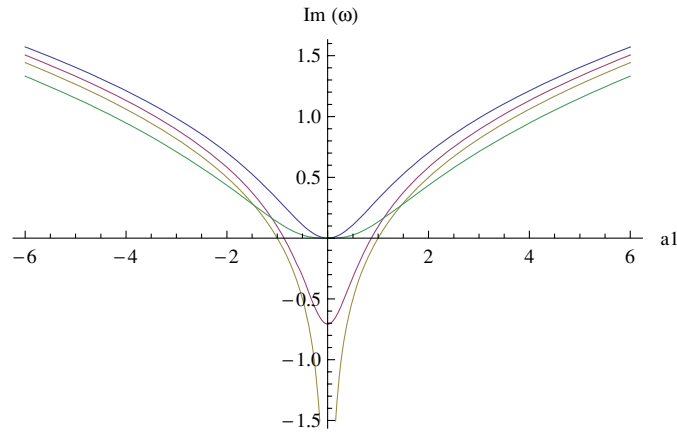


Figure 5. Variation of the positive value of $\text{Im}[\omega]$ against a_1 (keeping a_0 fixed). The different values of a_0 are 0 (blue), 0.5 (red), 1 (yellow) and 2 (green).

4. (Quantum) Field theoretic model

Now we come to the parent field theory action in (1 + 1)-dimensions, that is, the chiral Schwinger model (CSM)¹ [5, 6]:

$$S(A, \phi) = \int d^2x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + e(g^{\mu\nu} - \epsilon^{\mu\nu}) A_\nu (\partial_\mu \phi) + \frac{1}{2} a e^2 A_\mu A^\mu \right], \quad (18)$$

which explicitly turns out to be

$$S = \int d^2x \left[-\frac{1}{2} F_{01} F^{01} + \frac{1}{2} (\partial_0 \phi)^2 - \frac{1}{2} (\partial_1 \phi)^2 + e(\partial_0 \phi) A_0 + e\phi (\partial_0 A_1) + e(\partial_1 \phi) (A_0 - A_1) + \frac{1}{2} a e^2 (A_0^2 - A_1^2) \right], \quad (19)$$

¹ The CSM was derived from a theory of chiral fermions interacting with the electromagnetic field in (1 + 1)-dimensions $S = \int d^2x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} [i\partial + e\sqrt{\pi} \mathbf{A} (1 + i\gamma_5)] \psi \right]$, where $\gamma_5 = i \int \gamma^0 \gamma^1$ via bosonization.

where in (19) we have dropped a total time derivative term and in all the further calculations we will use expression (19). Note that in the limit of ignoring the spatial dependence (thereby dropping space derivatives), (19) reduces to (2) and (13) (when $a = 1$). The parameter a , known as the Jackiw–Rajaraman (JR) parameter, appears in the bosonized model (19) as a result of regularization ambiguity in evaluating the fermion determinant (see footnote 1). It is taken as a real number. Once again we will find that since the value of JR parameter a governs the constraint structure, it can alter the spectra although no qualitative changes are manifested in the Lagrangian or Hamiltonian of the model.

4.1. $a \neq 1$

From the momenta $\pi_0 = 0$, $\pi_1 = -(\partial_0 A_1 - \partial_1 A_0) + e\phi = -F_{01}$, $\pi = \partial_0 \phi + eA_0$, where $\{\phi(x), \pi(y)\} = \delta(x - y)$, $\{A_\mu(x), \pi^\nu(y)\} = g_\mu^\nu \delta(x - y)$, we obtain the Hamiltonian

$$H = \int dx \left(\frac{1}{2} \pi^2 + \frac{1}{2} \pi_1^2 + A_0 (\partial_1 \pi_1) + \frac{1}{2} (\partial_1 \phi)^2 - e\pi A_0 + \frac{1}{2} e^2 \phi^2 + \frac{1}{2} e^2 A_0^2 - \frac{1}{2} a e^2 (A_0^2 - A_1^2) + e(\partial_1 \phi) A_1 + e\phi \pi_1 \right) \quad (20)$$

and two constraints: $\psi_1 = \pi_0$ and $\psi_2 = \dot{\psi}_1 = \{\psi_1, H\} = (a - 1)e^2 A_0 + e\pi - \partial_1 \pi_1$ satisfying the algebra $\{\psi_1(x), \psi_2(y)\} = -(a - 1)e^2 \delta(x - y)$. Again for $a \neq 1$, the constraints are SCC. We eliminate A_0 and π_0 using the constraint equations and for rest of the variables, we see that the Dirac brackets remain the same. The reduced Hamiltonian is

$$H = \int dx \left(\frac{1}{2} \pi^2 + \frac{1}{2} \pi_1^2 + \frac{1}{2} e^2 \phi^2 + \frac{1}{2} (\partial_1 \phi)^2 + e(\partial_1 \phi) A_1 + e\phi \pi_1 + \frac{1}{2} a e^2 A_1^2 + \frac{1}{2e^2(a - 1)} ((\partial_1 \pi_1)^2 + e^2 \pi^2 - 2e\pi(\partial_1 \pi_1)) \right), \quad (21)$$

which yields the spectra, consisting of a Klein–Gordon scalar $\sigma = \pi_1 + e\phi$ and a harmonic mode $h = (\pi - eA_1) - \frac{1}{ae} \partial_1 (\pi_1 + e\phi)$,

$$\square \sigma + m^2 \sigma = 0, \quad \square h = 0, \quad m^2 = \frac{a^2 e^2}{a - 1}. \quad (22)$$

The theory is consistent for $a > 1$, otherwise there are tachyonic excitations. Note that h satisfies a higher derivative equation. These modes reduce to the previously computed spectra (12) when the space dependence is ignored.

4.2. $a = 1$

We directly put $a = 1$ in (19):

$$S = \int d^2x \left[-\frac{1}{2} F_{01} F^{01} + \frac{1}{2} (\partial_0 \phi)^2 - \frac{1}{2} (\partial_1 \phi)^2 + e(\partial_0 \phi) A_0 + e\phi (\partial_0 A_1) + e(\partial_1 \phi) (A_0 - A_1) + \frac{1}{2} e^2 (A_0^2 - A_1^2) \right], \quad (23)$$

and obtain the Hamiltonian

$$H = \int dx \left(\frac{1}{2} \pi^2 + \frac{1}{2} \pi_1^2 + \frac{1}{2} e^2 A_1^2 + \frac{1}{2} e^2 \phi^2 + \frac{1}{2} (\partial_1 \phi)^2 - e\pi A_0 + A_0 (\partial_1 \pi_1) + e(\partial_1 \phi) A_1 + e\phi \pi_1 \right). \quad (24)$$

Now there are three constraints:

$$\begin{aligned} \psi_1 &\equiv \pi_0, & \psi_2 &\equiv \dot{\psi}_1 = \{\psi_1, H\} = e\pi - (\partial_1\pi_1), \\ \psi_3 &\equiv \dot{\psi}_2 = \{\dot{\psi}_2, H\} = \pi_1 + e\phi. \end{aligned} \tag{25}$$

Using Dirac’s procedure [1], we find that ψ_1 is FCC and ψ_2, ψ_3 constitute an SCC pair. Here, the canonical bracket between the variables (ϕ, π) remains unchanged. Using the constraint equations, we have the Hamiltonian and a single massless mode $h = (\partial_0^2 + \partial_1^2)(\phi)$ respectively:

$$H = \int dx \left[\frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}(\partial_1\phi + eA_1)^2 \right], \quad \square h = 0, \tag{26}$$

which agree with the spectrum in the particle limit. This shows that the characteristic features associated with $a = 1$ in the previously analyzed model remain intact in the field theory as well.

Note that in both $a = 1$ and $a \neq 1$ cases, the *massless* modes satisfy higher derivative equations. In particular, h in (26) satisfy a fourth-order equation that is clearly reminiscent of the mode dynamics in the Pais–Uhlenbeck oscillator [4] that we have mentioned in section 1.

In passing, we make a generic comment. Note that the presence of gauge invariance fixes the value of a to $a = 1$. An analogous situation prevails in the bosonization of the vector Schwinger model [5, 6] which has gauge symmetry, and expectedly no arbitrary parameter appears in its bosonized version.

5. Cranking (particle) model and Maxwell–Chern–Simons–Proca (field) theory

We briefly mention the connection between the Cranking model [7], which is well known in nuclear physics and also recently studied [12] in the context of EP, and a widely studied relativistic field theory in $(2 + 1)$ -dimensions—the MCSP model [8–10]. We show that the connection is similar as the model (12) and CSM (19) considered in sections 3 and 4 if we drop the spatial derivatives. We consider the two-particle Lagrangian [9, 10]:

$$L = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + \frac{B}{2}(x_1x_2 - x_2x_1) - \frac{k}{2}(x_1^2 + x_2^2). \tag{27}$$

Using the momenta $p_i = \dot{x}_i - \frac{B}{2}\epsilon_{ij}x_j$, we find the Cranking model Hamiltonian [12]

$$H = \frac{1}{2} \left[p_1^2 + p_2^2 + \left(\frac{B^2}{4} + k \right) (x_1^2 + x_2^2) - B(x_1p_2 - x_2p_1) \right]. \tag{28}$$

We have scaled the masses to unity. After Bogoliubov transformation, the above Hamiltonian becomes diagonal where the energy eigenmodes are given by

$$\omega_{\pm}^2 = \frac{1}{2}(2k + B^2) \left[1 \pm \left(1 - \frac{4k^2}{(2k + B^2)^2} \right)^{\frac{1}{2}} \right]. \tag{29}$$

We emphasize that this model can be obtained from the MCSP model:

$$L_{\text{MCSP}} = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} + \frac{B}{2}\epsilon_{\mu\nu\lambda}(\partial^\mu A^\nu)A^\lambda + \frac{k}{2}A_\mu A^\mu, \tag{30}$$

by ignoring the space dependence and dropping the decoupled A_0 term. Taking account of the constraints in (30) and using the Dirac prescription [1], we find

$$H = \frac{1}{2}\pi_i^2 + \frac{1}{4}(A_{ij})^2 + \left(\frac{1}{2} + \frac{B^2}{8} \right) A_i^2 - \frac{B}{2}\epsilon_{ij}\pi_i A_j + \frac{1}{2} \left(\partial_i\pi_i + \frac{B}{2}\epsilon_{ij}(\partial_i A_j) \right)^2. \tag{31}$$

Applying the following nonlocal canonical transformations [8–10]:

$$A_i = \epsilon_{ij} \frac{\partial_j(Q_1 + Q_2)}{\sqrt{-\nabla^2}} + \frac{1}{2} \frac{\partial_i(P_1 - P_2)}{\sqrt{-\nabla^2}}, \quad \pi_i = \frac{1}{2} \epsilon_{ij} \frac{\partial_j(P_1 + P_2)}{\sqrt{-\nabla^2}} - \frac{\partial_i(Q_1 - Q_2)}{\sqrt{-\nabla^2}}, \quad (32)$$

the Hamiltonian becomes decoupled in the form

$$H = \left[\frac{1}{2} (P_1^2 + (\partial_i Q_1)^2 + M_1^2 Q_1^2) + \frac{1}{2} (P_2^2 + (\partial_i Q_2)^2 + M_2^2 Q_2^2) \right]. \quad (33)$$

Now M_1, M_2 are identical to ω_{\pm} in (29) and again ignoring spatial derivatives, the above becomes identical to spectra (29). This is our advertised correspondence.

6. Conclusion and future prospects

In this paper, we have demonstrated how a specific value of a single parameter (in the model, the JR parameter a) can influence the entire dynamical content of a model. This point in the parameter space is exceptional in the sense that the nature of the constraint structure is dictated by this specific point, i.e. at $a = 1$ the system has first class constraints, inducing local gauge invariance with a reduced number of physical degrees of freedom whereas away from $a = 1$ the system has only second class constraints with no additional invariance and so possessing a larger number of degrees of freedom. These results in the field theoretic chiral Schwinger model are not new, but the results in the corresponding finite dimensional particle model that we have formulated and studied are indeed new and interesting. We have explored the complex domain of the JR parameter a that reveals the existence of real and physically realizable energy values that were forbidden in the field theory context. We have also revealed the connection between two well-known discrete and field theoretic models, the Cranking model and the Maxwell–Chern–Simons–Proca model, studied in entirely different contexts. These works actually lead us to two interesting and topical areas, mentioned below, that we wish to pursue.

First of all, note that just now we have referred to the $a = 1$ point in the parameter point as exceptional for a specific reason. In fact, we believe that this point might be an interesting example of a novel type of exceptional point [11–14] in the context of non-Hermitian PT -symmetric quantum mechanics. The point $a = 1$ shares several features of the conventional exceptional point in the sense that the $a = 1$ point lies in the border of real and imaginary energy values (or non-unitary spectra in the case of field theory); there is an (apparent) singularity as $a \rightarrow 1$ (although the $a = 1$ point is not singular as further analysis shows) and there is a collapse of states (in this case, an infinite tower of states). (For qualitatively similar effects in the context of exceptional points, see [11, 12].) Furthermore, the Cranking model has already been studied in the context of exceptional points and PT -symmetric quantum mechanics and we have discussed here similar behavior for the discrete version of the chiral Schwinger model having real energy values for complex parameter a present in the Hamiltonian. These results can pave the way for the study of these features, i.e. PT -symmetry and exceptional points in the corresponding relativistic field theory models such as the chiral Schwinger model and Maxwell–Chern–Simons–Proca model.

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